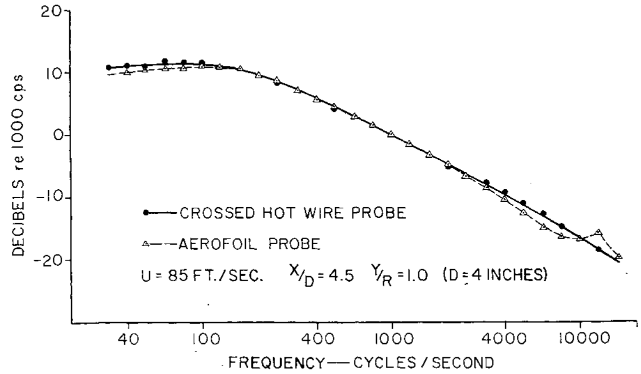

Fig. 3 Comparison of v component spectra in a round jet.

Superior durability and temporal stability should make possible measurements which were hitherto difficult with conventional crossed-wires (i.e., high velocities, dirty flows, hydrodynamic turbulence). Work is presently underway in the development of precise calibration techniques for the probe.

References

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Jet Damping of Symmetric Rockets

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Nomenclature

$\mathbf{H}$	= moment of momentum about center of mass
I_λ	= moment of inertia about the longitudinal axis
I_τ	= moment of inertia about any transverse axis
k	= radius of gyration
l	= distance from center of mass to jet along λ axis
m	= mass of rockets and contents at time t
$\mathbf{M}$, M	= moment, magnitude of moment
t	= time
XYZ	= inertial coordinates
α	= angle of attack
ρ	= radical distance to jet from λ axis
τ, ν, λ	= cross-spin coordinates
θ_p	= half angle of precession
θ, ϕ, ψ	= Euler angles
ω, Ω	= magnitude of angular velocity
$\mathbf{\Omega}$	= angular velocity of cross-spin coordinates

In recent years, the effect of jet damping on a spinning rocket has received considerable attention. Most studies have used the modified Eulerian dynamic equations[1] as the governing equations of motion. To find dynamic variables, such as transverse angular velocity and angle of attack, it is conventional to create complex expressions. There are exceptions,[2] but these have normally introduced complicated mathematical manipulations. Alternate equations of motion, derived herein, allow certain problems to be solved without the conventional complex variable substitution.

Received October 20, 1964.

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Assumptions

The equations of motion for a symmetric spinning body used by Nidey and Seames[3] are extended to include jet damping terms. In addition to symmetry, the conventional restrictions are assumed[4]: 1) no ejected particle is given an angular velocity relative to the rocket, 2) the angular momentum imparted to the particles is symmetric relative to the longitudinal axis, and 3) the center of mass always lies on the longitudinal axis. These assumptions give the general equation for rate of change of angular momentum:

$$\mathbf{M} = \dot{\mathbf{H}} + \text{rate of angular momentum transfer from the system} \quad (1)$$

Equations of Motion

Consider the cross-spin coordinate system τ, ν, λ shown in Fig. 1, where $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ are unit vectors parallel and normal to the axis of symmetry, respectively. Let $\boldsymbol{\tau}$ be oriented so that the angular velocity of the rocket has only two components ω_λ and ω_τ. The transverse component of angular velocity is sometimes called cross-spin. Because the rocket and the coordinate system share the transverse angular velocity, the angular velocity of the coordinate system $\mathbf{\Omega}$ is written as

$$\mathbf{\Omega} = \Omega_\lambda \boldsymbol{\lambda} + \omega_\tau \boldsymbol{\tau} \quad (2)$$

For a symmetric rigid body, ignoring jet damping, the moment of momentum and its time derivative are

$$\mathbf{H} = I_\lambda \omega_\lambda \boldsymbol{\lambda} + I_\tau \omega_\tau \boldsymbol{\tau} \quad (3)$$

$$\dot{\mathbf{H}} = I_\lambda \dot{\omega}_\lambda \boldsymbol{\lambda} + I_\tau \dot{\omega}_\tau \boldsymbol{\tau} + I_\lambda \omega_\lambda (\mathbf{\Omega} \times \boldsymbol{\lambda}) + I_\tau \omega_\tau (\mathbf{\Omega} \times \boldsymbol{\tau}) \quad (4)$$

thus

$$M_\lambda = I_\lambda \dot{\omega}_\lambda \quad (5)$$

$$M_\tau = I_\tau \dot{\omega}_\tau \quad (6)$$

and

$$M_\nu = I_\tau \omega_\tau \Omega_\lambda - I_\lambda \omega_\lambda \omega_\tau \quad (7)$$

where $\boldsymbol{\nu} \equiv \boldsymbol{\lambda} \times \boldsymbol{\tau}$. The rate of rotation of the cross-spin (transverse angular velocity) about the spin axis obtained by rearranging Eq. (7) is

$$\Omega_\lambda = \Omega_0 + M_\nu/(I_\tau \omega_\tau) \quad (8)$$

where

$$\Omega_0 \equiv (I_\lambda/I_\tau)\omega_\lambda$$

To include the effects of jet damping it is only necessary to add additional terms to the right side of Eqs. (5) and (6).

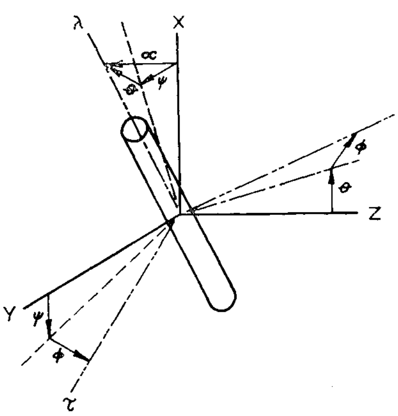

Fig. 1 Coordinates and Euler angles.

Using the same nomenclature as Ref. 4, the equations including jet damping are

$$M_\lambda = I_\lambda \dot{\omega} - \dot{m}\rho^2\omega_\lambda + \dot{I}_\lambda\omega_\lambda \tag{9}$$

$$M_\lambda = I_\tau\dot{\omega}_\tau - \dot{m}(l^2 + \tfrac{1}{2}\rho^2)\omega_\tau + \dot{I}_\tau\omega_\tau \tag{10}$$

or

$$M_\lambda = I_\lambda\dot{\omega}_\lambda - \dot{m}(\rho^2 - k_\lambda{}^2) - m(d/dt)k_\lambda{}^2 \tag{11}$$

$$M_\tau = I_\tau\dot{\omega}_\tau - \dot{m}(l^2 + \tfrac{1}{2}\rho^2 - k_\tau{}^2) - m(d/dt)\, k_\tau{}^2 \tag{12}$$

Note that Eq. (8) remains unchanged but Ω_0 is no longer a constant as in the case of a rigid body.

While an external moment is applied to the rocket, it is constrained to precess with the angular velocity Ω given by (2) and half angle

$$\theta p = \arctan \omega_\tau/\Omega_\lambda \tag{13}$$

In general, $\mathbf{\Omega}$ is not fixed in space but will rotate about an inertial reference at a varying rate and with a varying inclination. Equations (2) and (13) reduce to free-body equations when $M_\nu = 0$.

Examples

The transverse angular velocity of a moment-free rocket with an initial spin $\omega_\lambda(0)$ will be found without resorting to complex variable notation. So that the results obtained here can be compared with known solutions,[3] we assume that the fuel burns in such a manner that the variations in k_λ and k_τ are negligible.

Equation (11) becomes

$$\int \frac{d\omega_\lambda}{\omega_\lambda} = \left(\frac{\rho^2}{k_\lambda{}^2} - 1\right)\int\frac{dm}{m} \tag{14}$$

which yields

$$\frac{\omega_\lambda}{\omega_\lambda(0)} = \left(\frac{m}{m_0}\right)^{(\rho^2/k_\lambda{}^2)-1} \tag{15}$$

In a similar manner, from Eq. (12), we obtain

$$\int\frac{d\omega_\tau}{\omega_\tau} = \left(\frac{l^2 + \frac{1}{2}\rho^2 - k_\tau{}^2}{k_\tau{}^2}\right)\int\frac{dm}{m} \tag{16}$$

which yields

$$\frac{\omega_\tau}{\omega_\tau(0)} = \left(\frac{m}{m_0}\right)^{(l^2+\frac{1}{2}\rho^2-k_\tau{}^2)/k_\tau{}^2} \tag{17}$$

The speed that ω_τ revolves around the spin axis is found from Eq. (8) to be

$$\frac{\Omega_\lambda}{\Omega_\lambda(0)} = \left(\frac{m}{m_0}\right)^{(\rho^2/k_\tau{}^2)-1} \tag{18}$$

since $I_\tau/I_\lambda = \text{const.}$

Because differential time did not appear explicitly in the integrals, the resulting Eqs. (15–18) are not restricted to any particular time variation of m/m_0. When $M = m_0 - m't$ is substituted into (17) and (18), the results are the same as given in Ref. 4.

The variations in the angle-of-attack α can be found from the pitch and yaw angles (Fig. 1) as given by expressions for the Euler angles. Using the aeronautical convention,

$$\dot{\phi} = \Omega_\lambda + \dot{\psi}\sin\theta \tag{19}$$

$$\dot{\theta} = \omega_\tau\cos\phi \tag{20}$$

$$\dot{\psi}\cos\theta = \omega_\tau\sin\phi \tag{21}$$

For small angle approximations, Eqs. (19–21) become

$$\dot{\phi} = \Omega_\lambda \tag{22}$$

$$\dot{\theta} = \omega_\tau\cos\phi \tag{23}$$

$$\dot{\psi} = \omega_\tau\sin\phi \tag{24}$$

and the angle of attack is

$$\alpha = [\theta^2 + \psi^2]^{1/2} \tag{25}$$

To solve Eqs. (22–25), the ratio m/m_0, which is a function of time, must be integrated so that the results are not as general as in the case of cross-spin. Equations (23) and (24) can be integrated directly or can be added in quadrature to obtain

$$\dot{\theta} + i\dot{\psi} = \omega_\tau e^{i\phi} \tag{26}$$

Thus,

$$\theta - \theta_0 = Re\int_0^t \omega_\tau\, e^{i\phi}\, dt \tag{27}$$

$$\psi - \psi_0 = Im\int_0^t \omega_\tau\, e^{i\phi}\, dt \tag{28}$$

where

$$\phi = \int_0^t \Omega_\lambda\, dt + \phi_0$$

For initial moment-free body $\phi_0 = \pi/2$, $\theta_0 = \alpha_0 = \theta p(0)$, and $\psi_0 = 0$ if XYZ are chosen so that the X axis coincides with the angular momentum vector and the XZ plane contains the spin axis of the rocket at $t = 0$.

For the second example, consider the same rocket with a constant spin-up moment M_λ applied to the rocket. We first observe that the result for cross-spin, Eq. (17), is still valid. That is, the magnitude of the cross-spin is independent of the spin-up moment. Note also that the angular rate Ω_λ varies directly with ω_λ, which is found by integrating

$$M_\lambda = mk_\lambda{}^2\dot{\omega}_\lambda - \dot{m}(\rho - k_\lambda{}^2)\omega_\lambda \tag{29}$$

Put in standard form,

$$\dot{\omega}_\lambda + P(t)\omega_\lambda = Q(t) \tag{30}$$

where

$$P(t) = \frac{\dot{m}}{m}\left(\frac{\rho - k_\lambda{}^2}{k_\lambda{}^2}\right) \tag{31}$$

$$Q(t) = M_\lambda/mk_\lambda{}^2 \tag{32}$$

When $m(t)$ is known, ω_λ is found by the conventional method used to solve a first-order, linear differential equation with variable coefficients.

Conclusion

It has been shown by example that it is possible to solve certain problems without creating complex variables. When the foregoing equations of motion are used in conjunction with the Euler dynamic equations or in their place, solutions to the complicated problems of spinning bodies may be found more readily. The physical implications of Eq. (8), which aid in understanding the gyroscopic behavior of spinning bodies, are not repeated since they were discussed in a previous comment.[3]

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